5. Vul'fson, I. I. and Kolovskii, M. Z., Nonlinear Problems of Machine Dynamics. Leningrad, "Machinostroenie", 1968.
6. Rapoport, I. M., Some Asymptotic Methods in the Theory of Differential Equations. Kiev, Izd. Akad. Nauk USSR, 1954.
7. Jahnke, E. and Emde, F., Tables of Functions With Formulas and Curves. Moscow, Fizmatgiz, 1959.
8. Kamke, E., Handbook of Ordinary Differential Equations, 3rd ed., Moscow. "Nauka", 1965

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STEADYSTATE MOTIONS IN AUTONOMOUS SYSTEMS WITH A DEVIATING ARGUMENT

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L. D. AKULENKO
(Moscow)
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A rotating-phase autonomous system with a deviating argument is investigated. A scheme of successive approximations for the exact solution over an infinite time interval is constructed; sufficient conditions for the existence of a steadystate solution are derived. Such systems occur frequently in the theory of nonlinear vibrational-rotational motions in systems whose parameters vary within a narrow range.

Let us construct the stationary, i.e. steadystate, solutions of a real system of the form

$$dE / dt = \varepsilon f (E, E_\tau, \psi, \psi_\tau, \varepsilon) \qquad (E_\tau = E (t - \tau))$$
$$d\psi / dt = \omega (E, E_\tau) + \varepsilon F (E, E_\tau, \psi, \psi_\tau, \varepsilon) \qquad (\psi_\tau = \psi (t - \tau)) \tag{1}$$

Here $t \in (-\infty, \infty)$ is the time, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ a small parameter, E a vector variable whose values lie in some neighborhood of the point E_0^*, $\psi \in (-\infty, \infty)$ the scalar phase, and $\tau \in (-\infty, \infty)$ a constant.

We can construct the solution by the method of successive approximations [1], making use of the fact that if system (1) has a solution $E(t), \psi(t)$ for all t, then it also has a family of solutions $E(t + \theta)$, $\psi(t + \theta)$, where θ is an arbitrary constant. The value of the phase ψ can therefore be chosen arbitrarily for some instant t_0. For example, we can set it equal to zero in order to simplify our expressions. To avoid secular terms in system (1) we introduce the new independent variable s such that

$$t - t_0 = s (1 + \varepsilon h), \qquad \tau = \varphi (1 + \varepsilon h)$$

This yields the system

$$dE / ds = \varepsilon (1 + \varepsilon h) f (E, E_\varphi, \psi, \psi_\varphi, \varepsilon)$$
$$d\psi / ds = (1 + \varepsilon h) [\omega (E, E_\varphi) + \varepsilon F (E, E_\varphi, \psi, \psi_\varphi, \varepsilon)]$$

where h is some constant which we choose in such a way that the solution of the perturbed system in s has the "unperturbed" period T_0.

Assuming that the functions f, ω have partial derivatives with respect to all their arguments and that these derivatives together with F satisfy the Lipschitz condition in the above domain, we make the substitutions

$$E = E_0 + \varepsilon x, \quad \psi = \omega_0 s + \theta + \varepsilon y \qquad (E_0, \theta = \text{const})$$

to obtain the system

$$\frac{dx}{ds} = f(E_0, E_0, \omega_0 s, \omega_0(s-\varphi), 0) + \varepsilon\Big[hf_0 + \left(\frac{\partial f}{\partial E}\right)_0 x + \left(\frac{\partial f}{\partial E_\varphi}\right)_0 x_\varphi + \left(\frac{\partial f}{\partial \psi}\right)_0 y +$$
$$+ \left(\frac{\partial f}{\partial \psi_\varphi}\right)_0 y_\varphi + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 + f^*(s, \varphi, h, x, x_\varphi, y, y_\varphi, \varepsilon)\Big] \tag{2}$$
$$\frac{dy}{ds} = h\omega(E_0, E_0) + \left(\frac{\partial \omega}{\partial E}\right)_0 x + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 x_\varphi + F_0 + F^*(s, \varphi, h, x, x_\varphi, y, y_\varphi, \varepsilon)$$
$$T_0 = 2\pi / \omega(E_0, E_0)$$

Here x, y are unknown periodic functions of s of period T_0 ; f^*,F^* are unknown functions which vanish identically for $\varepsilon = 0$. The zeroth approximations of the functions x, y can be obtained from system (2) by setting $\varepsilon = 0$ and ensuring that $y = 0$ for $s = 0$. Hence,

$$x_0(s) = a_0 + \int_0^s f_0\, ds_1 \equiv a_0 + x_0^*(s) \qquad (a_0 = \text{const})$$

$$y_0(s) = \left[h_0\omega_0 + \left(\frac{\partial \omega}{\partial E}\right)_0 a_0 + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 a_0\right] s + \int_0^s \left[\left(\frac{\partial \omega}{\partial E}\right)_0 x_0^*(s_1) + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 x_0^*(s_1 - \varphi) + F_0\right] ds_1$$

The vector function x_0 is periodic if

$$R(E_0, \varphi) = \int_0^{T_0} f(E_0, E_0, \omega_0 s, \omega_0(s-\varphi), 0)\, ds \equiv T_0 \langle f_0 \rangle = 0 \tag{3}$$

and the resulting nonlinear system of equations is the defining system for the vector E_0. Let $E_0^* = E_0(\varphi)$ be a root of this system (3) which belongs to the permissible domain. Similarly, the function y_0 is periodic if we set

$$h_0 = -\frac{1}{\omega_0}\left\langle \left(\frac{\partial \omega}{\partial E}\right)_0 x_0^* + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 x_{0\varphi}^* + F_0 \right\rangle - \left[\left(\frac{\partial \omega}{\partial E}\right)_0 + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0\right] \frac{a_0}{\omega_0}$$

The periodic function y_0 is therefore defined completely, while x_0 is defined only to within the constant vector a_0.

The first approximation system is (2) in which the functions f^* and F^* have been set equal to zero. On substituting the zeroth approximation into the equations for x_1 we obtain

$$x_1(s) = a_1 + x_0^*(s) + \varepsilon\left[h_0 x_0^* + \int_0^s \left(\left(\frac{\partial f}{\partial E}\right)_0 + \left(\frac{\partial f}{\partial E_\varphi}\right)_0\right) a_0\, ds_1 + \int_0^s f_1(s_1, \varphi)\, ds_1\right]$$
$$\left(f_1 = \left(\frac{\partial f}{\partial E}\right)_0 x_0^* + \left(\frac{\partial f}{\partial E_\varphi}\right)_0 x_{0\varphi}^* + \left(\frac{\partial f}{\partial \psi}\right)_0 y_0 + \left(\frac{\partial f}{\partial \psi_\varphi}\right)_0 y_{0\varphi} + \left(\frac{\partial f}{\partial \varepsilon}\right)_0\right) \tag{4}$$

This implies that

$$(\partial R / \partial E_0^*)\, a_0 = -T_0 \langle f_1 \rangle$$

This system of linear equations in the vector a_0 is uniquely solvable if $\det(\partial R/\partial E_0^*) \neq 0$ We are assuming that this is, in fact, the case. Further, the expression

$$y_1(s) = \left[h_1\omega_0 + \left(\frac{\partial \omega}{\partial E}\right)_0 a_1 + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 a_1\right] s +$$
$$+ \int_0^s \left[\left(\frac{\partial \omega}{\partial E}\right)_0 x_1^* + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 x_{1\varphi}^* + F_0 + F^*(s_1, \varphi, h_0, x_0, x_{0\varphi}, y_0, y_{0\varphi}, \varepsilon)\right] ds_1$$

implies that

$$h_1 = -\frac{1}{\omega_0}\langle F_1 \rangle - \left[\left(\frac{\partial \omega}{\partial E}\right)_0 + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0\right] \frac{a_1}{\omega_0}$$
$$\left(F_1(s, \varphi, h_0, \varepsilon) = \left(\frac{\partial \omega}{\partial E}\right)_0 x_1^* + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 x_{1\varphi}^* + F_0 + F_0^*\right)$$

The subsequent approximations are obtainable from the general scheme

$$\frac{dx_l}{ds} = f_0 + \varepsilon\left[h_{l-1} f_0 + \left(\frac{\partial f}{\partial E}\right)_0 x_{l-1} + \left(\frac{\partial f}{\partial E_\varphi}\right)_0 x_{l-1,\,\varphi} + \right.$$
$$\left. + \left(\frac{\partial f}{\partial \psi}\right)_0 y_{l-1} + \left(\frac{\partial f}{\partial \psi_\varphi}\right)_0 y_{l-1,\,\varphi} + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 + f_{l-1}^*\right] \tag{5}$$
$$\frac{dy_l}{ds} = h_l\omega_0 + \left(\frac{\partial \omega}{\partial E}\right)_0 x_l + \left(\frac{\partial \omega}{\partial E_\varphi}\right)_0 x_{l\varphi} + F_0 + F_{l-1}^* \qquad (l \geqslant 2)$$

Substituting the functions x_1 and y_1 into the vector equation for x_2, we obtain a formula similar to (4).

$$x_2(s) = a_2 + x_0^*(s) + \varepsilon\left[h_1 x_0^* + \int_0^s \left(\left(\frac{\partial f}{\partial E}\right)_0 a_1 + \left(\frac{\partial f}{\partial E_\varphi}\right)_0 a_1 + f_2\right) ds_1\right]$$
$$\left(f_2(s, \varphi, a_1, \varepsilon) = \left(\frac{\partial f}{\partial E}\right)_0 x_1^* + \left(\frac{\partial f}{\partial E_\varphi}\right)_0 x_{1\varphi}^* + \left(\frac{\partial f}{\partial \psi}\right)_0 y_1 + \left(\frac{\partial f}{\partial \psi_\varphi}\right)_0 y_{1\varphi} + \right.$$
$$\left. + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 + f^*(s, \varphi, h_1, x_1, x_{1\varphi}, y_1, y_{1\varphi}, \varepsilon) \qquad (f_2|_{\varepsilon=0} = f_1)\right)$$

Thus, the vector equation in a_1 obtainable from the periodicity condition for x_2 has a real root $a_1(\varphi, \varepsilon)$ for sufficiently small ε. The second-approximation system therefore yields the complete first approximation of the vector function x and the second approximation for y. It can be shown by induction on the basis of the implicit function theorem that system (5) enables us to find any approximation in powers of ε of functions x, y, h (where $h = H(\varphi, \varepsilon)$) periodic in s, φ. Solving the equation

$$h = H(\tau(1 + \varepsilon h), \varepsilon) \qquad (h = h(\tau, \varepsilon)) \tag{6}$$

for h, we obtain explicit expressions for the required unknowns,

$$E(t, \tau, \varepsilon) = E_0(\varphi) + \varepsilon x(s, \varphi, \varepsilon), \qquad \psi(t, \tau, \varepsilon) = \omega(E_0(\varphi), E_0(\varphi))s + \varepsilon y(s, \varphi, \varepsilon)$$
$$(s = (t - t_0)[1 + \varepsilon h(\tau, \varepsilon)]^{-1}, \quad \varphi = \tau[1 + \varepsilon h(\tau, \varepsilon)])$$

We can construct the solution of Eq. (6) by successive approximations according to the scheme

$$h_j = H(\tau(1 + \varepsilon h_{j-1}), \varepsilon) \qquad (j \geqslant 1,\ h_0 = H(\tau, 0))$$

Many autonomous problems of the theory of nonlinear vibrational-rotational motions are reducible to systems of the type (1). We refer, specifically, to systems with one degree of freedom whose parameters vary within a narrow range, to the autonomous analog of the system investigated in [2], et al. The proposed small-parameter method [1] is a more direct way of dealing with systems with a deviating argument and has certain other advantages over the averaging schemes of [3], where $t \sim 1/\varepsilon$.

BIBLIOGRAPHY

1. Malkin, I. G., Some Problems of the Theory of Nonlinear Vibrations. Moscow, Gostekhizdat, 1956.
2. Akulenko, L. D., Steady-state periodic and rotational motions in perturbed, significantly nonlinear and almost conservative systems with one degree of freedom, in the case of an arbitrary constant deviation of the argument. PMM Vol. 32, №2, 1968.
3. Volosov, V. M., Medvedev, G. N. and Morgunov, B. I., Using an averaging method to solve certain systems of differential equations with a deviating argument. Vest. Mosk. Univ., Ser. Fiz. Astron., №4, 1965

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